Theorem 75 Prove that grad $(f+g) =$ grad $f +$ grad g.

Proof:- grad $(f+g) = \vec{\nabla}(f+g)$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](f+g)$$

$$= \left[\frac{\partial}{\partial x}(f+g), \frac{\partial}{\partial y}(f+g), \frac{\partial}{\partial z}(f+g)\right]$$

$$= \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}, \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}, \frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] + \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] f + \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g$$

$$= \vec{\nabla} f + \vec{\nabla} g$$

grad $(f+g) =$ grad $f +$ grad g. proved.

Theorem 76 Prove that grad $(fg) = f$ grad $g + g$ grad f

Proof:- grad$(fg) = \vec{\nabla}(fg)$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](fg) = \left[\frac{\partial}{\partial x}(fg), \frac{\partial}{\partial y}(fg), \frac{\partial}{\partial z}(fg)\right]$$

$$= \left[f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g, f\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}g, f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g\right]$$

$$= \left[f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right] + \left[\frac{\partial f}{\partial x}g, \frac{\partial f}{\partial y}g, \frac{\partial f}{\partial z}g\right]$$

$$= f\left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right] + g\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= f\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]g + g\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]f$$

$$= f\,\vec{\nabla}g + g\,\vec{\nabla}f$$

grad $(fg) = f$ grad $g + g$ grad f. Proved

(Q12/78) Prove that

(i) $\text{grad}(fgh) = gh\,\text{grad}\, f + hf\,\text{grad}\, g + fg\,\text{grad}\, h$.

Sol:- $\text{grad}(fgh) = \text{grad}[f(gh)]$

$= f\,\text{grad}(gh) + gh\,\text{grad}\, f$.

$= f[g\,\text{grad}\, h + h\,\text{grad}\, g] + gh\,\text{grad}\, f$

$= fg\,\text{grad}\, h + fh\,\text{grad}\, g + gh\,\text{grad}\, f$

Ch-05

rewritting in the reverse order we get

$\text{grad}(fgh) = gh\,\text{grad}\, f + hf\,\text{grad}\, g + fg\,\text{grad}\, h$.

ii) $\text{grad}\left(\frac{f}{g}\right) = \frac{g\,\text{grad}\, f - f\,\text{grad}\, g}{g^2}$.

Sol:- $\text{grad}\left(\frac{f}{g}\right) = \text{grad}\, f(g)^{-1}$

$= f\,\text{grad}\, g^{-1} + g^{-1}\,\text{grad}\, f$

$= f\,\vec{\nabla}.g^{-1} + \frac{1}{g}\,\text{grad}\, f$

$= f\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g^{-1} + \frac{1}{g}\,\text{grad}\, f$

$= f\left[\frac{\partial}{\partial x} g^{-1}, \frac{\partial}{\partial y} g^{-1}, \frac{\partial}{\partial z} g^{-1}\right] + \frac{1}{g}\,\text{grad}\, f$

$= f\left[-g^{-2}\frac{\partial g}{\partial x}, -g^{-2}\frac{\partial g}{\partial y}, -g^{-2}\frac{\partial g}{\partial z}\right] + \frac{1}{g}\,\text{grad}\, f$

$= f\left(-\frac{1}{g^2}\right)\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] g + \frac{1}{g}\,\text{grad}\, f$

$= -\frac{f}{g^2}\vec{\nabla} g + \frac{1}{g}\,\text{grad}\, f$

$= -\frac{f}{g^2}\,\text{grad}\, g + \frac{1}{g}\,\text{grad}\, f$

$= \frac{-f\,\text{grad}\, g + g\,\text{grad}\, f}{g^2}$

$\text{grad}\left(\frac{f}{g}\right) = \frac{g\,\text{grad}\, f - f\,\text{grad}\, g}{g^2}$ Proved

Ex-4/77 If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then evaluate grad r^n.

Sol: $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = |\vec{r}| = (x^2+y^2+z^2)^{1/2}$$

$$r^n = \left[(x^2+y^2+z^2)^{1/2}\right]^n = (x^2+y^2+z^2)^{n/2}$$

$$\text{grad } r^n = \text{grad } (x^2+y^2+z^2)^{n/2} = \vec{\nabla}(x^2+y^2+z^2)^{n/2}$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right](x^2+y^2+z^2)^{n/2}$$

$$= \left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}, \frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2}, \frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}\right]$$

$$= \left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x), \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2y), \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2z)\right]$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}[x, y, z]$$

$$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= n\left[(x^2+y^2+z^2)^{1/2}\right]^{n-2}\vec{r}$$

$$= n\, r^{n-2}\,\vec{r} \qquad \because (x^2+y^2+z^2)^{1/2} = r$$

$$\boxed{\text{grad } r^n = n\, r^{n-2}\,\vec{r}}$$ proved

$\frac{11}{79}$ prove $\operatorname{div}(c\vec{f}) = c \operatorname{div} f$, c is constant.

Sol:- let $\vec{f} = [f_1, f_2, f_3]$

$$\operatorname{div}(c\vec{f}) = \vec{\nabla} \cdot c[f_1, f_2, f_3]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [cf_1, cf_2, cf_3]$$

$$= \frac{\partial}{\partial x} cf_1 + \frac{\partial}{\partial y} cf_2 + \frac{\partial}{\partial z} cf_3$$

$$= c\left[\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}\right]$$

$$= c\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [f_1, f_2, f_3]$$

$\operatorname{div}(c\vec{f}) = c \operatorname{div}\vec{f}$. proved.

(iii/79) prove $div(\phi \vec{F}) = \phi \, div F + grad \, \phi \cdot F$.

let $\vec{F} = [f_1, f_2, f_3]$

$$div(\phi \vec{F}) = \vec{\nabla} \cdot \phi \vec{F} = \vec{\nabla} \cdot \phi [f_1, f_2, f_3]$$

$$= \vec{\nabla} \cdot [\phi f_1, \phi f_2, \phi f_3] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [\phi f_1, \phi f_2, \phi f_3]$$

$$= \frac{\partial}{\partial x}(\phi f_1) + \frac{\partial}{\partial y}(\phi f_2) + \frac{\partial}{\partial z}(\phi f_3)$$

$$= \phi \frac{\partial f_1}{\partial x} + \frac{\partial \phi}{\partial x} f_1 + \phi \frac{\partial f_2}{\partial y} + \frac{\partial}{\partial y} \phi f_2 + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial z} f_3$$

$$= \phi \frac{\partial f_1}{\partial x} + \phi \frac{\partial f_2}{\partial y} + \phi \frac{\partial f_3}{\partial z} + \frac{\partial \phi}{\partial x} f_1 + \frac{\partial \phi}{\partial x} f_1 + \frac{\partial \phi}{\partial y} f_2 + \frac{\partial \phi}{\partial z} f_3$$

Ch-05

$$div(\phi \vec{F}) = \phi \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) + \left[\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right] \cdot [f_1, f_2, f_3]$$

$$= \phi \vec{\nabla} \cdot \vec{F} + \vec{\nabla} \phi \cdot \vec{F}$$

$$div(\phi \vec{F}) = \phi \, div \vec{F} + grad \, \phi \cdot \vec{F}$$. proved.

CURL OF A VECTOR

Let $\vec{F} = [F_1, F_2, F_3]$ be a vector, Then

$\vec{\nabla} \times \vec{F}$ is called curl of $\vec{F}$ and denoted by

Curl $\vec{F}$.

$$\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Ex 83) Calculate Curl $\vec{F}$ at $(1,1,1)$ if

$$F(x,y,z) = x^2 y\,\vec{i} + (4xz + y^2)\vec{j} + (5z^2 + xy^2)\vec{k}.$$

Sol:- $\text{Curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 4xz + y^2 & 5z^2 + xy^2 \end{vmatrix}$

By. R_1

$$\text{Curl}\,\vec{F} = \vec{i}\left[\frac{\partial}{\partial y}(5z^2 + xy^2) - \frac{\partial}{\partial z}(4xz + y^2)\right]$$

$$+\vec{j}\left[\frac{\partial}{\partial z}(x^2-y)-\frac{\partial}{\partial x}(5z^2+xy^2)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(4xz+y^2)-\frac{\partial}{\partial y}(x^2y)\right]$$

$$=\vec{i}\left[(0+2xy)-(4x+0)\right]+\vec{j}\left[0-(0+y^2)\right]$$

$$+\vec{k}\left[(4z+0)-x^2\right]$$

at $p(1,1,1)$.

$$(\text{Curl }\vec{F})_p=(2\cdot 1-4\cdot 1)\vec{i}+\vec{j}((-1)^2)+\vec{k}(4\cdot 1-$$

$$(\text{Curl }\vec{F})_p=-2\vec{i}-\vec{j}+3\vec{k}\quad \underline{\text{Ans}}$$

Calculate Curl $\vec{F}$ if

(Q1/84) $\vec{F} = (y^2+z^2)\vec{i} + (z^2+x^2)\vec{j} + (x^2+y^2)\vec{k}$.

Sol:- Curl $\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix}$

$= \vec{i}\left[\frac{\partial}{\partial y}(x^2+y^2) - \frac{\partial}{\partial z}(z^2+x^2)\right] + \vec{j}\left[\frac{\partial}{\partial z}(y^2+z^2) - \frac{\partial}{\partial x}(x^2+y^2)\right]$
$+ \vec{k}\left[\frac{\partial}{\partial x}(z^2+x^2) - \frac{\partial}{\partial y}(y^2+z^2)\right]$

$= \vec{i}[2y-2z] + \vec{j}[2z-2x] + \vec{k}[2x-2y]$

Curl $\vec{F} = 2(y-z)\vec{i} + 2(z-x)\vec{j} + 2(x-y)\vec{k}$. Ans

$\frac{Q2}{84}$ $\vec{F} = (x^2+y^2+z^2)^{-\frac{1}{2}}(yz\vec{i} + zx\vec{j} + xy\vec{K})$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2+z^2)^{-\frac{1}{2}}yz & (x^2+y^2+z^2)^{-\frac{1}{2}}zx & (x^2+y^2+z^2)^{-\frac{1}{2}}xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}xy - \frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}yz\right]$$

$$+\vec{j}\left[\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-\frac{1}{2}}yz + \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}xy\right]$$

$$+\vec{K}\left[\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}}zx - \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-\frac{1}{2}}yz\right]$$

$$= \vec{i}\left[x\{(x^2+y^2+z^2)^{-\frac{1}{2}}.1 + y(-\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}(2y)\}\right.$$
$$\left. - y\{(x^2+y^2+z^2)^{-\frac{1}{2}}.1 + \cancel{x}(-\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}2\cancel{z}\}\right]$$

$$+\vec{j}\left[x\{(x^2+y^2+z^2)^{-\frac{1}{2}}.1 + z(-\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}.2z\right.$$
$$\left. - y\{(x^2+y^2+z^2)^{-\frac{1}{2}}.1 + x(-\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}2x\}\right]$$

$$+\vec{K}\left[z\{(x^2+y^2+z^2)^{-\frac{1}{2}}.1 + x(-\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)\right.$$
$$\left. - z\{(x^2+y^2+z^2)^{-\frac{1}{2}}.1 + y(-\tfrac{1}{2})(x^2+y^2+z^2)^{-\frac{3}{2}}2y\right]$$

$$= \vec{i}(x^2+y^2+z^2)^{-\frac{3}{2}}\left[x\{x^2+y^2+z^2-y^2\} - y\{x^2+y^2+\cancel{z^2}-\cancel{z^2}\}\right]$$
$$+\vec{j}(x^2+y^2+z^2)^{-\frac{3}{2}}\left[y\{x^2+y^2+z^2-z^2\} - y\{\cancel{x^2}+y^2+z^2-\cancel{x^2}\}\right]$$
$$+\vec{K}(x^2+y^2+z^2)^{-\frac{3}{2}}\left[z\{x^2+y^2+z^2\} - x^2\{-z\{x^2+\cancel{y^2}+z^2-\cancel{y^2}\}\right]$$

Ch-05

(39)

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[x(\cancel{x^2}+z^2-\cancel{x^2}-y^2)\vec{i} + y(x^2+\cancel{y^2}-\cancel{y^2}-z^2)\vec{j}\right.$$
$$\left. + z(y^2+z^2-x^2-z^2)\vec{K}\right]$$

$$= (x^2+y^2+z^2)^{-\frac{3}{2}}\left[x(z^2-y^2)\vec{i} + y(x^2-z^2)\vec{J} + z(y^2-x^2)\vec{K}\right]$$

Ans

Q3/84 $\vec{F} = yz\cos x\,\vec{i} + zx\cos y\,\vec{j} + xy\cos z\,\vec{k}$

$$\text{Curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz\cos x & zx\cos y & xy\cos z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy\cos z) - \frac{\partial}{\partial z}(zx\cos y)\right] + \vec{j}\left[\frac{\partial}{\partial z}(yz\cos x) - \frac{\partial}{\partial x}(xy\cos z)\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(zx\cos y) - \frac{\partial}{\partial y}(yz\cos x)\right]$$

$$= \vec{i}\,[x\cos z - x\cos y] + \vec{j}\,[y\cos x - y\cos z]$$
$$+ \vec{k}\,[z\cos y - z\cos x].$$

$$\text{Curl}\,\vec{F} = x(\cos z - \cos y)\vec{i} + y(\cos x - \cos z)\vec{j} + z(\cos y - \cos x)\vec{k}$$

Ans